2025
NHT

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Write your **student number** in the boxes above. **Letter**

Chenghao Wu solutions

Mathematical Methods
Examination 1

Question and Answer Book

VCE (NHT) Examination – Tuesday 27 May 2025

- Reading time is **15 minutes**: 10.30 am to 10.45 am
- Writing time is **1 hour**: 10.45 am to 11.45 am

Materials supplied

- Question and Answer Book of 16 pages
- Formula Sheet

Instructions

Students are **not** permitted to bring any technology (calculators or software), or notes of any kind, into the examination room.

Students are **not** permitted to bring mobile phones and/or any unauthorised electronic devices into the examination room.

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8 questions (40 marks)	2–13

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Instructions

- Answer **all** questions in the spaces provided.
- Write your responses in English.
- In all questions where a numerical answer is required, an **exact** value must be given unless otherwise specified.
- In questions where more than one mark is available, appropriate working **must** be shown.
- Unless otherwise indicated, the diagrams in this book are **not** drawn to scale.

Question 1 (4 marks)

a. Let $y = \frac{\log_e(x)}{x^3}$.

Find and simplify $\frac{dy}{dx}$. 2 marks

$$\frac{dy}{dx} = \frac{x^3 \cdot \frac{1}{x} - \ln(x)\, 3x^2}{x^6}$$

$$= \frac{x^2 - \ln(x) \cdot 3x^2}{x^6}$$

$$= \frac{1 - 3\ln(x)}{x^4}$$

b. Let $f(x) = \sin(\pi e^{3x})$.

Evaluate $f'(0)$. 2 marks

$$f'(x) = 3\pi e^{3x} \cos(\pi e^{3x})$$

$$f'(0) = 3\pi e^{0} \cos(\pi e^{0})$$

$$= 3\pi \cos(\pi)$$

$$= -3\pi$$

N H T

Question 2 (3 marks)

Let

$$g : R \to R,\ g(x) = ax^3 + bx^2 + c \text{ where } a, b, c \in R$$

Could $a = 0$, $b = 0$?

and

$$h : R \to R,\ h(x) = \frac{x^2}{4}$$

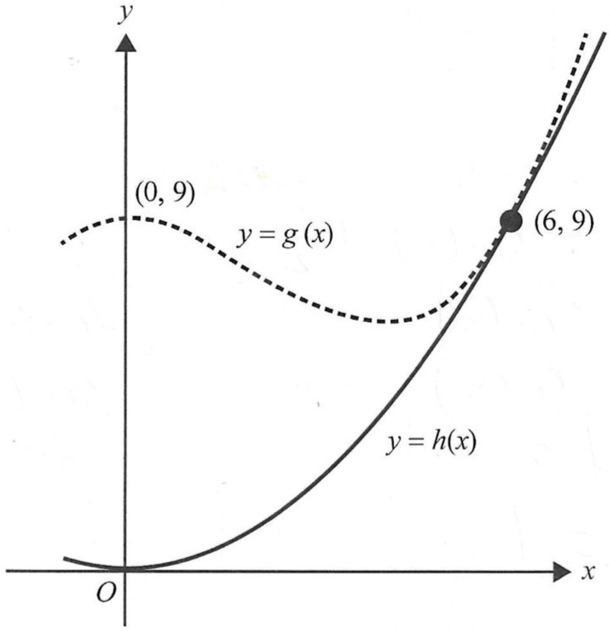
The graph of $y = g(x)$ passes through the point $(0, 9)$.

The graphs of $y = g(x)$ and $y = h(x)$ pass through the point $(6, 9)$ and have the same gradient at this point. formatting?

The graphs of $y = g(x)$ and $y = h(x)$ are shown below.

$36 \times 6 = 216$

$36 \times 3 = 108$



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Find the values of a, b and c.

$9 = g(0) \Rightarrow 9 = c$

$g(x) = ax^3 + bx^2 + 9$

$9 = g(6) \Rightarrow 9 = 216a + 36b + 9$

$0 = 216a + 36b$

$0 = 6a + b$ (1)

~~$9 = h(6) \Rightarrow 9 = \frac{36}{4}$~~

$g'(x) = 3ax^2 + 2bx$

$h'(x) = \frac{x}{2}$

$h'(6) = \frac{6}{2} = 3$

$g'(6) = 3 \Rightarrow 3 = 3a \cdot 36 + 2b \cdot 6$

$3 = 108a + 12b$

$1 = 36a + 4b$

$\frac{1}{4} = 9a + b$ (2)

(2) − (1): $\frac{1}{4} = 3a$

$a = \frac{1}{12}$

Sub into (1):

$0 = \frac{6}{12} + b$

$b = -\frac{1}{2}$

$$\begin{cases} a = \frac{1}{12} \\ b = -\frac{1}{2} \\ c = 9 \end{cases}$$

Question 3 (7 marks)

Let $f : D \to R,\ f(x) = \dfrac{1}{2}\tan(x) + \dfrac{1}{2}$, where D is the maximal domain.

a. Find the general solution for $f(x) = 0$. 2 marks

$-\frac{1}{2} = \frac{1}{2}\tan(x)$

$-1 = \tan(x)$

S | A

T | C

$x = \pi - \frac{\pi}{4},\ -\frac{\pi}{4},\ 3\pi - \frac{\pi}{4},\ 2\pi - \frac{\pi}{4}, \ldots$

$x = n\pi - \frac{\pi}{4},\ n \in \mathbb{Z}$

b. Let $g : R \to R,\ g(x) = \sin(x) + \dfrac{1}{2}$.

i. Find all values of x such that $f(x) = g(x)$ for $x \in (-\pi, \pi)$. 2 marks

$\frac{1}{2}\tan(x) + \frac{1}{2} = \sin(x) + \frac{1}{2}$

$\frac{1}{2}\tan(x) = \sin(x)$

$\frac{1}{2}\frac{\sin(x)}{\cos(x)} = \sin(x)$

if $\sin(x) \neq 0$

$\frac{1}{2}\frac{1}{\cos(x)} = 1$

S | A

T | C

$\frac{1}{2} = \cos(x)$

$x = \frac{\pi}{3},\ \frac{-\pi}{3}$

If $\sin(x) = 0$

$\frac{1}{2}\tan(x) = 0$

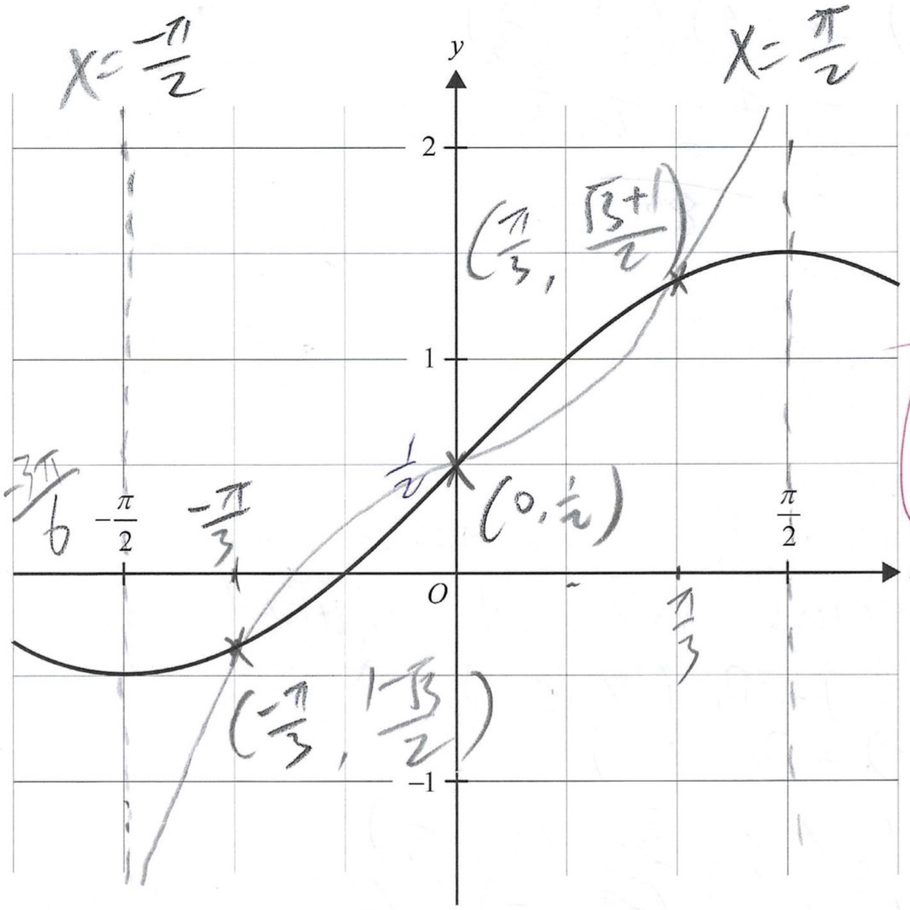
$\tan(x) = 0$

$x = 0$

$x = \frac{\pi}{3},\ \frac{-\pi}{3},\ 0$

ii. The graph of $y = g(x)$ is shown below.

On the set of axes below, sketch the graph of $y = f(x)$ over the interval $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$, labelling all points where $f(x) = g(x)$ with their coordinates and all asymptotes with their equations. **3 marks**



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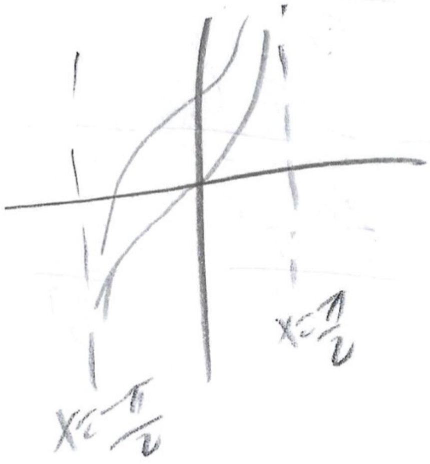

$f(0) = g(0) = \frac{1}{2}$

$f\left(-\frac{\pi}{3}\right) = g\left(-\frac{\pi}{3}\right) = \sin\left(-\frac{\pi}{3}\right) + \frac{1}{2}$

$= -\frac{\sqrt{3}}{2} + \frac{1}{2}$

$= \frac{1-\sqrt{3}}{2}$

$f\left(\frac{\pi}{3}\right) = g\left(\frac{\pi}{3}\right) = \sin\left(\frac{\pi}{3}\right) + \frac{1}{2}$

$= \frac{\sqrt{3}}{2} + \frac{1}{2}$

Question 4 (5 marks)

$\frac{1}{3}$

An orchard contains a large number of apples. One third of the apples are bruised.

4

a. In a random sample of four apples, find the probability that exactly three are bruised. 2 marks

Let $X \sim Bi(4, \frac{1}{3})$

$Pr(X=3) = \binom{4}{3}\left(\frac{1}{3}\right)^3\left(\frac{2}{3}\right)^1$

$= 4 \cdot \frac{1}{27} \frac{2}{3}$

$= \frac{8}{81}$

b. Let $\hat{P}$ be the random variable representing the proportion of bruised apples in random samples of 10 apples from the orchard.

Find $\Pr\left(\hat{P} \leq \frac{1}{10}\right)$.

Give your answer in the form $\frac{2^m}{3^n}$, where m and n are positive integers. 3 marks

$\hat{p} = \frac{1}{10}$ means $10 \times \frac{1}{10} = 1$ bruised apple.

$Pr(\hat{p} \leq \frac{1}{10}) = Pr(X \leq 1)$

$= Pr(X=0) + Pr(X=1)$

where $X \sim Bi(10, \frac{1}{3})$

$Pr(X=0) = \binom{10}{0} \left(\frac{1}{3}\right)^0 \left(\frac{2}{3}\right)^{10} = \left(\frac{2}{3}\right)^{10}$

$Pr(X=1) = \binom{10}{1}\left(\frac{1}{3}\right)^1\left(\frac{2}{3}\right)^9 = 10 \cdot \frac{2^9}{3^{10}}$

$Pr(\hat{p} \leq \frac{1}{10}) = \frac{2^{10}}{3^{10}} + \frac{10 \cdot 2^9}{3^{10}} = \frac{2^9(10+2)}{3^{10}}$

$= \frac{2^9 \cdot (12)}{3^{10}} = \frac{2^9 \cdot 3 \cdot 4}{3^9 \cdot 3} = \frac{2^9 \cdot 2^2}{3^9} = \frac{2^{11}}{3^9}$

Question 5 (4 marks)

Let $f: R \to R, f(x) = e^{2x}$ and $g: R \to R, g(x) = e^{x} + 2$.

a. Find the x-value of the point of intersection of the graphs of $y = f(x)$ and $y = g(x)$. 2 marks

$e^{2x} = e^{x} + 2$

let $u = e^{x}$

$u^{2} = u + 2$

$u^{2} - u - 2 = 0$

$(u-2)(u+1) = 0$

$u = 2$ or $u = -1$

$e^{x} = 2$ or $e^{x} = -1$

$e^{x} = -1$ is impossible

$e^{x} = 2$ is only solution

$x = \log_e(2)$

b. Find the x-value where the graphs of $y = f(x)$ and $y = g(x - 1)$ have equal gradients. 2 marks

let $h(x) = g(x-1) = e^{x-1} + 2$

$f'(x) = 2e^{2x}$

$h'(x) = e^{x-1}$

$2e^{2x} = e^{x-1}$

$2e^{x}e^{x} = e^{x}e^{-1}$

$2e \cdot e^{x} \cdot e^{x} = e^{x}$

$2e \cdot e^{x} = 1$

$e^{x} = \frac{1}{2e}$

$x = \log_e\left(\frac{1}{2e}\right)$

Question 6 (4 marks)

A high jumper has up to three attempts to jump over a bar at a specific height.

Once one attempt is successful, the high jumper does not make another attempt at this height.

At this height, let

- A be the event that the high jumper's first attempt is successful
- B be the event that the high jumper's second attempt is successful
- C be the event that the high jumper's third attempt is successful.

$\Pr(A) = 0.6$ and $\Pr(B \mid A') = 0.5$

a. Let N be a discrete random variable representing the number of attempts the high jumper makes at this height.

i. Complete the table below for the probability distribution of N. 1 mark

n	1	2	3
$\Pr(N = n)$	0.6	0.2	0.2

ii. Find $\mathrm{E}(N)$, the mean number of attempts the high jumper makes at this height. 1 mark

$0.6 + 0.4 + 0.6$

$= 1.6$

b. The high jumper was not successful on the first attempt at this height.

Find $\Pr(C \mid B')$, given that the high jumper has a 0.7 probability of making a successful jump on either their second or third attempt. 2 marks

Refer to diagram on next page (page 9)

~~$0.4 \times 0.5 + 0.4 \times 0.5 \times x = 0.7$~~

~~$0.2 + 0.2x = 0.5$~~

~~$0.2x = 0.5$~~

~~$x = \frac{5}{2}$~~

~~$\Pr(C \mid B') = \frac{5}{2}$~~

Answer also on next page

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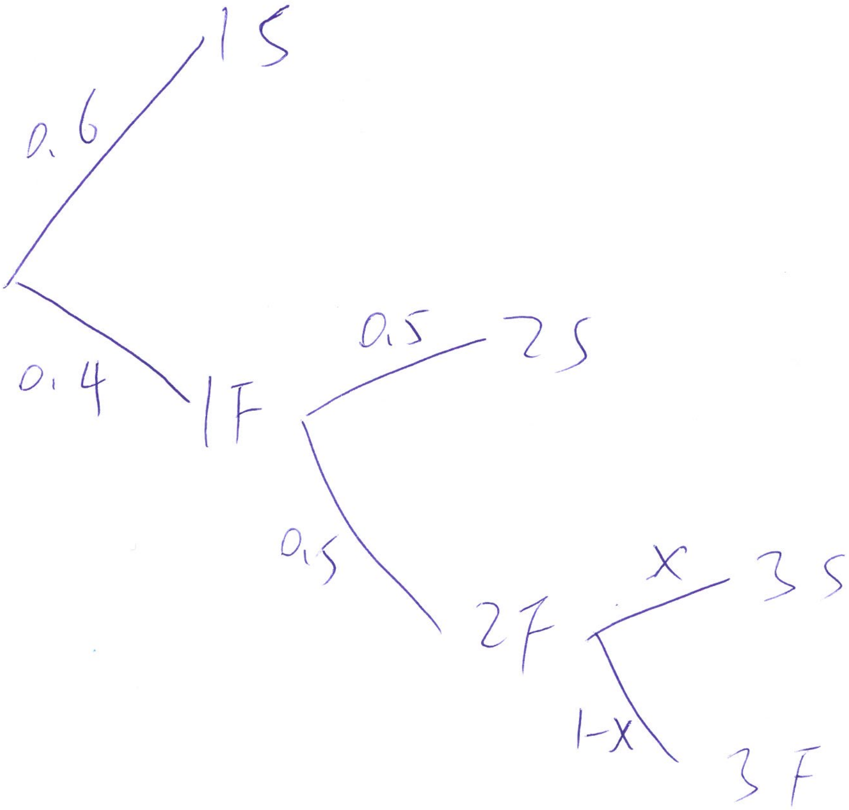

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b. $0.5 + 0.5x = 0.7$

$0.5x = 0.2$

$x = \frac{2}{5}$

$Pr(C \mid B') = \frac{2}{5}$

Question 7 (7 marks)

Let $f: R \to R, f(x) = x^2 + bx - 6$, where $b \in R$.

$x = \frac{-b \pm \sqrt{b^2-4ac}}{2a}$

a. Find the values of b for which the two solutions of $f(x) = 0$ have a difference of 5. 2 marks

$x^2+bx-6=0$

$x = \frac{-b \pm \sqrt{b^2-4\cdot 1\cdot(-6)}}{2}$

$= \frac{-b \pm \sqrt{b^2+24}}{2}$

$\frac{-b+\sqrt{b^2+24}}{2} - \frac{-b-\sqrt{b^2+24}}{2} = 5$

$\frac{-b+\sqrt{b^2+24}+b+\sqrt{b^2+24}}{2} = 5$

$\sqrt{b^2+24} = 5$

$b^2+24 = 25$

$b^2 = 1$

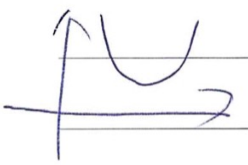
$b = \pm 1$

$b=1$ or $b=-1$

b. Let $b < 0$.

i. Explain why an inverse function does not exist when f is restricted to the domain $(0, \infty)$. 1 mark

Turning point is at $x = \frac{-b}{2}$. If $b<0$, then $\frac{-b}{2} > 0$, so ~~turnin~~ turning point has positive x-value. It is also a positive parabola. Shape is



because turing point is to the right of $x=0$, in the domain $(0, \infty)$. ~~f~~ f is not one-to-one function.

ii. Find a rule for the inverse function of f when f is restricted to the domain $(-\infty, 0)$. 2 marks

let $y = x^2+bx-6$

swap x and y

$x = y^2+by-6$

$x+6 = y^2+by$

$x+6 = \left(y+\frac{b}{2}\right)^2 - \left(\frac{b}{2}\right)^2$

$\left(y+\frac{b}{2}\right)^2 = x+6+\frac{b^2}{4}$

$y+\frac{b}{2} = \pm\sqrt{x+6+\frac{b^2}{4}}$

~~$\text{dom } f^{-1} = \text{ran } f$~~

$\text{ran } f^{-1} = \text{dom } f = (-\infty, 0)$

$y = -\sqrt{x+6+\frac{b^2}{4}} - \frac{b}{2}$

$f^{-1}(x) = -\sqrt{x+6+\frac{b^2}{4}} - \frac{b}{2}$

c. Let g be a function with maximal domain and the rule $g(x) = \log_e(-x-2)$. $= \ln(-(x+2))$

Find the value of b for which the maximal domain of the composite function $g \circ f$ is $(-2, 2)$. 2 marks

$g \circ f(x) = \ln(-(x^2+bx-6+2))$

$= \ln(-(x^2+bx-4))$

$= \ln(-x^2-bx+4)$

~~turning point of~~ let $h(x) = -x^2-bx+4$

~~Turning point of $y=h(x)$ is $x = \frac{b}{-2}$~~

~~$h\left(\frac{b}{-2}\right) = -\frac{b^2}{4} - b\left(\frac{-b}{2}\right) + 4$~~

~~$= -\frac{b^2}{4} + \frac{b^2}{2} + 4$~~

$h(x) = 0$

$-x^2-bx+4=0$

$x^2+bx-4=0$

$x = \dfrac{-b \pm \sqrt{b^2 - 4\cdot 1\cdot(-4)}}{2}$

$= \dfrac{-b \pm \sqrt{b^2+16}}{2}$

$\dfrac{-b + \sqrt{b^2+16}}{2} = 2$

$-b + \sqrt{b^2+16} = 4$

$b^2+16 = (4+b)^2$

$b^2+16 = 16+8b+b^2$

$0 = 8b$

$b = 0$

Question 8 (6 marks)

Let $f: [0, \infty) \to R$, $f(x) = \dfrac{x}{2\sqrt{x+k}}$, where k is a positive real number.

a. Consider the particular case where $k = 6$.

i. Show that $\dfrac{d}{dx}\left(x\sqrt{x+6}\right) = \sqrt{x+6} + \dfrac{x}{2\sqrt{x+6}}$. 1 mark

$$\frac{d}{dx}\left(x\sqrt{x+6}\right) = x \cdot \frac{1}{2}(x+6)^{-\frac{1}{2}} + \sqrt{x+6} \cdot 1$$
$$= \frac{x}{2\sqrt{x+6}} + \sqrt{x+6}$$
$$= \sqrt{x+6} + \frac{x}{2\sqrt{x+6}}$$

ii. Hence, or otherwise, evaluate the definite integral $\int_0^6 f(x)\,dx$. 3 marks

$$\int \sqrt{x+6} + \frac{x}{2\sqrt{x+6}}\,dx = x\sqrt{x+6} + c$$
$$\int \sqrt{x+6}\,dx + \int \frac{x}{2\sqrt{x+6}}\,dx = x\sqrt{x+6} + c$$
$$\frac{2}{3}(x+6)^{3/2} + \int f(x)\,dx = x\sqrt{x+6} + c$$
$$\int_0^6 f(x)\,dx = \left[x\sqrt{x+6} - \frac{2}{3}(x+6)^{3/2}\right]_0^6$$
$$= 6\cdot\sqrt{12} - \frac{2}{3}\cdot 12^{3/2} - 0 + \frac{2}{3}\cdot 6^{3/2}$$
$$= 6\cdot\sqrt{12} - \cancel{2}\,2\cdot 3^{-1}\cdot 3^{3/2}\cdot 4^{3/2} + 2\cdot 3^{-1}\cdot 2^{3/2}\cdot 3^{3/2}$$
$$= 6\sqrt{12} - 2^4\,3^{1/2} + 2^{5/2}\cdot 3^{1/2}$$
$$= 2\cdot 3\cdot 3^{1/2}\cdot 4^{1/2} - 2^4 3^{1/2} + 2^{5/2}3^{1/2}$$
$$= 2^2\cdot 3^{3/2} - 2^4\,3^{1/2} + 2^{5/2}\,3^{1/2}$$
$$= 3^{1/2}\left(2^2\cdot 3 - 2^4 + 2^{5/2}\right)$$
$$= 3^{1/2}\left(12 - 16 + 4\sqrt{2}\right)$$

why so ugly?

$\frac{\sqrt{2}}{2}$ $\frac{\sqrt{2}}{\sqrt{2}}$ $\frac{\sqrt{2}}{\sqrt{2}}$ $\frac{\sqrt{2}}{\sqrt{2}}$

b. Find $\int_0^k f(x)\,dx$ in the form $\frac{a-\sqrt{a}}{b}k\sqrt{k}$ where a and b are positive integers. 2 marks

$$\frac{d}{dx}\left(x\sqrt{x+k}\right) = x\,\frac{1}{2}(x+k)^{-\frac{1}{2}} + \sqrt{x+k}\cdot 1$$

$$= \frac{x}{2\sqrt{x+k}} + \sqrt{x+k}$$

$$\int f(x)\,dx + \int \sqrt{x+k}\,dx = x\sqrt{x+k} + c$$

$$\int f(x)\,dx + \frac{2}{3}(x+k)^{3/2} = x\sqrt{x+k}$$

$$\int_0^k f(x)\,dx = \left[x\sqrt{x+k} - \frac{2}{3}(x+k)^{3/2}\right]_0^k$$

$$= k\sqrt{2k} - \frac{2}{3}(2k)^{3/2} - 0 + \frac{2}{3}k^{3/2}$$

$$= k\,2^{1/2}k^{1/2} - 2\cdot 3^{-1}\cdot 2^{3/2}k^{3/2} + 2\cdot 3^{-1}\cdot k^{3/2}$$

$$= 2^{1/2}k^{3/2} - \frac{2^{5/2}}{3}k^{3/2} + \frac{2}{3}k^{3/2}$$

$$= k^{3/2}\left(\frac{3\cdot 2^{1/2}}{3} - \frac{2^{5/2}}{3} + \frac{2}{3}\right)$$

$$= k\sqrt{k}\left(\frac{3\sqrt{2} - 4\sqrt{2} + 2}{3}\right)$$

$$= \frac{2-\sqrt{2}}{3}k\sqrt{k}$$

$$\rightarrow \sqrt{3}\left(-4 + 4\sqrt{2}\right)$$

$$= \sqrt{3}\,4\left(\sqrt{2} - 1\right)$$

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End of examination questions

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