bigWU

Mathematical Methods Units 3 & 4

Written Examination 1 (Technology-free) 2025

Question and Answer Book

Student Name ______Solutions______

Teacher Name ________________

Reading Time: 15 minutes
Writing Time: 60 minutes

The total number of marks available is 40

Materials Supplied
- Question and Answer Book
- Formula Sheet

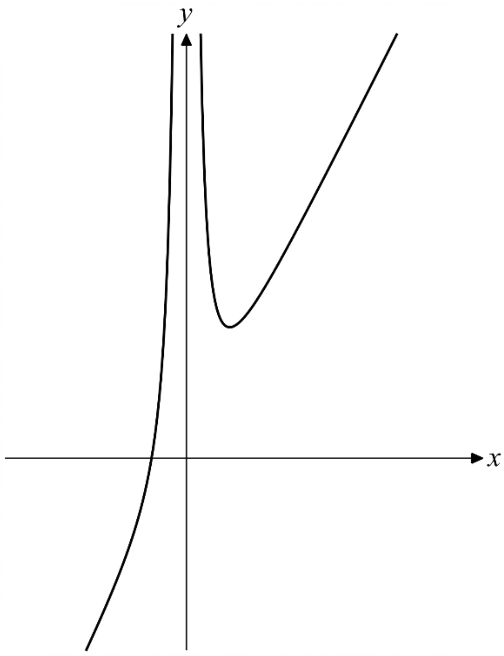
Instructions
- Students are **not** permitted to bring any technology (calculators or software), or notes of any kind, into the examination room.
- Students are **not** permitted to bring mobile phones and/or any unauthorised electronic devices into the examination room.

Instructions

- Answer all questions in the spaces provided.
- Write your responses in English.
- In questions where a numerical answer is required, an exact value must be given unless otherwise specified.
- In questions where more than one mark is available, appropriate working must be shown.
- Unless otherwise indicated, the diagrams in this book are not drawn to scale.

Question 1 (9 marks)

Consider function $f(x) = \frac{1}{x^2} + 2x$ defined over its maximal domain. The graph of $y = f(x)$ is shown in the diagram below.



a. Find the coordinates of the stationary point on the graph of $y = f(x)$.

$f'(x) = -2x^{-3} + 2$ [1 mark – derivative function]

$0 = \frac{-2}{x^3} + 2$

$\frac{2}{x^3} = 2$

$x^3 = 1$

$x = 1, y = 3$

$(1, 3)$ [1 mark – answer]

2 marks

b. Newton's method is used to approximate the solution to the equation $f(x) = 0$.

i. If $x_0 = -2$, find the value of x_1.

$f(-2) = \frac{1}{4} - 4 = \frac{1}{4} - \frac{16}{4} = -\frac{15}{4}$

$f'(-2) = \frac{2}{8} + 2 = \frac{1}{4} + \frac{8}{4} = \frac{9}{4}$

$x_1 = -2 - \left(-\frac{15}{4}\right) \cdot \frac{4}{9} = -2 + \frac{15}{9} = -\frac{18}{9} + \frac{15}{9} = -\frac{3}{9} = -\frac{1}{3}$

1 mark

ii. State the values of x that are not allowed to be x_0.

$x = 0$ and $x = 1$.

1 mark

c. The bisection method is used to approximate the solution to the equation $f(x) = 0$. An algorithm written in pseudocode for the bisection method is shown below.

```
define f(x):
      return 1/x^2 + 2x

a ← ■
b ← ■
m ← (a+b)/2
count ← 0
while b − a > 2 × 0.0001
      if f(a) × f(m) < 0 then
            b ← m
      else
            a ← m
      end if
      [redacted]
      count ← count + 1
end while
print m, count
```

The initial values of a and b are not yet chosen, but it is assumed that $b > a$.
A line of code between "`end if`" and "$count \leftarrow count + 1$" is hidden.

i. Could the initial value of b be 0? If no, why? If yes, why and how should the initial value of a be chosen? Justify your answer with reasoning.

Yes, as long as the initial value of a is chosen such that $f(a) < 0$. [1 mark]

This is because $f(b)$ is never evaluated in the algorithm, so even though $f(0)$ is undefined, the algorithm would not encounter $f(0)$. [1 mark]

2 marks

ii. What is the hidden line of code between "`end if`" and "$count \leftarrow count + 1$"?

$m \leftarrow \frac{a+b}{2}$

1 mark

d. Consider function $g(x) = -\frac{1}{x^2} + 2x$ defined over its maximal domain. Find the sequence of transformations that transform the graph of $y = f(x)$ to the graph of $y = g(x)$. Justify your answer with reasoning.

$-f(x) = -\left(\frac{1}{x^2} + 2x\right) = -\frac{1}{x^2} - 2x = h(x)$

$h(-x) = -\frac{1}{(-x)^2} + 2x = -\frac{1}{x^2} + 2x = g(x)$ [1 mark – algebraic working]

A reflection in the x-axis.

A reflection in the y-axis. [1 mark – answer, order can be different]

2 marks

Question 2 (10 marks)
A medical laboratory is testing a new rapid diagnostic test for a viral infection. The test has a 90% accuracy rate (probability of correctly identifying whether someone has the virus or not). Assume test results are independent of each other.

a. In a clinical trial, five patients are each tested once.

i. Find the probability that exactly two tests have correct results.

$\binom{5}{2} \times 0.9^2 \times 0.1^3 = 10 \times 0.81 \times 0.001 = 0.81 \times 0.01 = 0.0081$

[1 mark – evidence of binomial formula]
[1 mark – answer]

2 marks

ii. Find the most likely number of tests which have a correct result, correct to the nearest whole number.

$5 \times 0.9 = 4.5 \approx 5$

1 mark

b. The laboratory decides to test patients one at a time until they get their first incorrect result, at which point they stop testing.

i. Find the probability that the first incorrect result occurs on the 3rd test.

$0.9^2 \times 0.1 = 0.81 \times 0.1 = 0.081$

1 mark

ii. Given that the first two tests were all correct, find the probability that the first incorrect result occurs within the next two tests (i.e. on the 3rd or 4th test).

Hint: draw a tree diagram.

$\Pr(\text{3rd or 4th incorrect}|\text{first two correct}) = \frac{\Pr(\text{3rd or 4th incorrect} \cap \text{first two correct})}{\Pr(\text{first two correct})}$

$= \frac{0.9^2 \times 0.1 + 0.9^3 \times 0.1}{0.9^2} = 0.1 + 0.9 \times 0.1 = 0.1 + 0.09 = 0.19$

[1 mark – evidence of conditional probability formula]

[1 mark – answer]

2 marks

c. A 'positive' test result is defined to indicate the patient is infected. A 'negative' test result is defined to indicate the patient is healthy.

To improve reliability, the laboratory implements a 'double-testing' protocol: each patient is tested twice, and the patient is diagnosed as infected only if both tests are positive. Otherwise, the patient is diagnosed as healthy.

Suppose a group contains six patients who are indeed infected and four patients who are indeed healthy. One patient is randomly selected from the group and double-tested.

i. Find the probability that an indeed infected patient is diagnosed as infected.

Hint: draw a tree diagram.

$0.6 \times 0.9^2 = 0.6 \times 0.81 = 0.486$

1 mark

ii. Find the probability that the randomly selected patient is diagnosed as healthy.

$1 - 0.486 - 0.4 \times 0.1^2 = 0.514 - 0.004 = 0.510$

1 mark

iii. Given that the randomly selected patient is diagnosed as infected, find the probability that they are indeed infected.

$\Pr(\text{indeed infected}|\text{infected diagnosis}) = \frac{\Pr(\text{indeed infected} \cap \text{infected diagnosis})}{\Pr(\text{infected diagnosis})}$

$= \frac{0.486}{0.486+0.004} = \frac{0.486}{0.490} = \frac{243}{245}$

[1 mark – recycle one of the answers from part i and part ii and put correctly into formula]
[1 mark – answer]

2 marks

Question 3 (11 marks)

a. Find $\frac{d}{dx}(\cos(x)\log_e(\cos(x)))$, where e is Euler's number.

$\cos(x)\frac{-\sin(x)}{\cos(x)} + \log_e(\cos(x))\,(-\sin(x))$ [1 mark – evidence of product rule]

$= -\sin(x) - \sin(x)\log_e(\cos(x))$ [1 mark – answer]

2 marks

b. Hence, show that $\int \sin(x)\log_e(\cos(x))\;dx = -\cos(x)\log_e(\cos(x)) + \cos(x) + c$, where c is a constant.

From part a, $-\int \sin(x)\;dx - \int \sin(x)\log_e(\cos(x))\;dx = \cos(x)\log_e(\cos(x))$.

$-\cos(x)\log_e(\cos(x)) - \int \sin(x)\;dx = \int \sin(x)\log_e(\cos(x))\;dx$

$\int \sin(x)\log_e(\cos(x))\;dx = -\cos(x)\log_e(\cos(x)) + \cos(x) + c$

1 mark

c. Hence, show that $\int_0^{\frac{\pi}{3}} \sin(x)\log_e(\cos(x))\;dx = \frac{1}{2}\log_e(2) - \frac{1}{2}$.

$\int_0^{\frac{\pi}{3}} \sin(x)\log_e(\cos(x))\;dx = [-\cos(x)\log_e(\cos(x)) + \cos(x)]_0^{\frac{\pi}{3}}$ [1 mark – working]

$= -\cos\left(\frac{\pi}{3}\right)\log_e\left(\cos\left(\frac{\pi}{3}\right)\right) + \cos\left(\frac{\pi}{3}\right) + \cos(0)\log_e(\cos(0)) - \cos(0)$

$= -\frac{1}{2}\log_e\left(\frac{1}{2}\right) + \frac{1}{2} + \log_e(1) - 1$ [1 mark – working]

$= -\frac{1}{2}\log_e(2^{-1}) - \frac{1}{2}$

$= \frac{1}{2}\log_e(2) - \frac{1}{2}$

2 marks

Let $f(x) = \sin(x)\log_e(\cos(x))$.

d. On the graph of $y = f(x)$, there is an asymptote at $x = \frac{\pi}{2}$. Explain why there is an asymptote at $x = \frac{\pi}{2}$.

If we try to substitute $x = \frac{\pi}{2}$ into f, we get $f\left(\frac{\pi}{2}\right) = \sin\left(\frac{\pi}{2}\right)\log_e\left(\cos\left(\frac{\pi}{2}\right)\right) = \log_e(0)$.

However, 0 is not allowed to be an input into any logarithmic function, hence $f\left(\frac{\pi}{2}\right)$ is undefined, hence there is an asymptote at $x = \frac{\pi}{2}$.

1 mark

e. For $x \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$, by considering individual components of $f(x)$, prove that $f(-x) = -f(x)$.

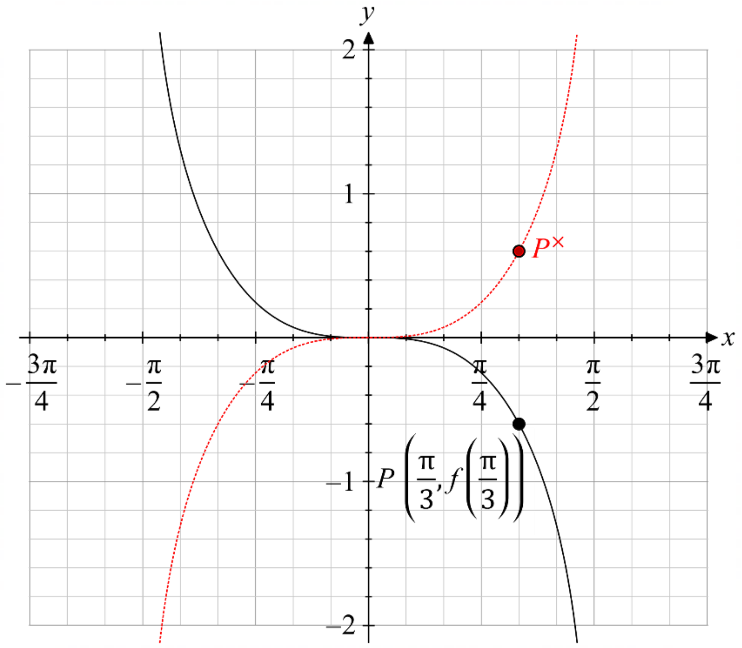
$f(-x) = \sin(-x)\log_e(\cos(-x))$

By symmetry, $\sin(-x) = -\sin(x)$ and $\cos(-x) = \cos(x)$.

Hence, $f(-x) = -\sin(x)\log_e(\cos(x)) = -f(x)$

1 mark

The graph of $y = f(x)$ for $x \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$ is shown in the diagram below. The point P has coordinates $\left(\frac{\pi}{3}, f\left(\frac{\pi}{3}\right)\right)$.



f. On the diagram above, clearly draw the graph of $y = -f(x)$ for $x \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$. Label the image of point P as P^*.

2 marks

[1 mark – graph symmetrical and correct scale], [1 mark – P^* correctly labelled]

g. Hence, find the area bounded by the curves of $y = f(x)$ and $y = -f(x)$, and the lines $x = -\frac{\pi}{3}$ and $x = \frac{\pi}{3}$.

$\left(-\frac{1}{2}\log_e(2) + \frac{1}{2}\right) \times 4 = -2\log_e(2) + 2$

[1 mark – negate answer from question c]

[1 mark – answer]

2 marks

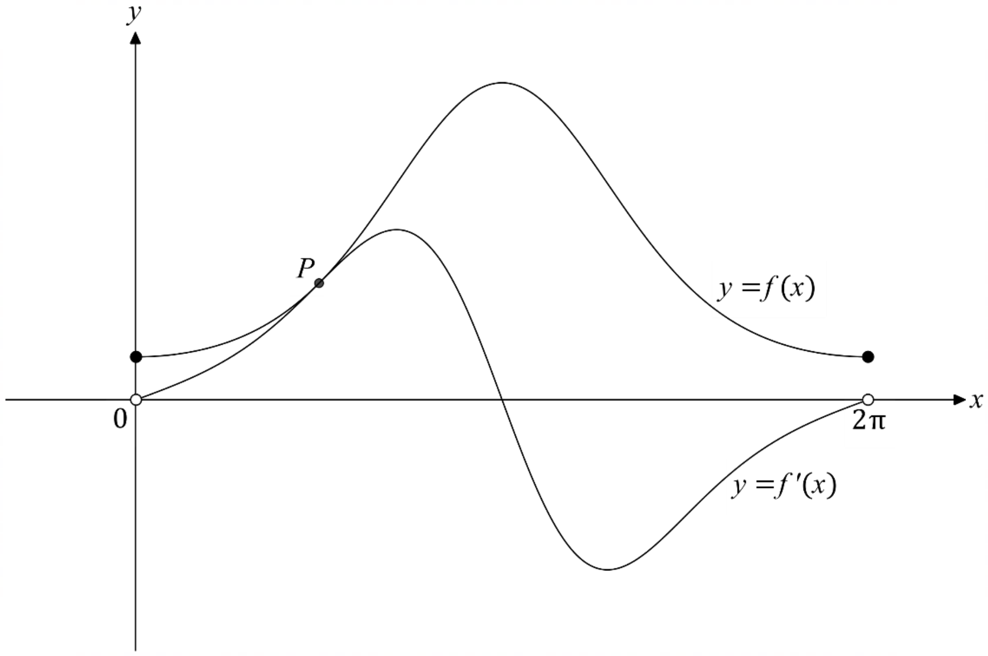
Question 4 (10 marks)
Consider function

$$f: [0, 2\pi] \rightarrow \mathbb{R}, f(x) = e^{-\cos(x)}$$

and its derivative function

$$f': (0, 2\pi) \rightarrow \mathbb{R}, f'(x) = e^{-\cos(x)} \sin(x)$$

where e is Euler's number.
The diagram below shows the graphs of $y = f(x)$ and $y = f'(x)$. The two graphs touch at point P.



a. Determine the x-coordinate of point P.

$e^{-\cos(x)} = e^{-\cos(x)} \sin(x)$

$1 = \sin(x)$ [1 mark – any of the two lines above]

$x = \frac{\pi}{2}$ [1 mark – answer]

2 marks

b. Show that $f''(x) = e^{-\cos(x)}(\cos(x) + 1 - \cos^2(x))$. Clearly state any trigonometric identity you use.

$\frac{d}{dx}\left(e^{-\cos(x)}\sin(x)\right) = e^{-\cos(x)}\cos(x) + \sin(x) \cdot \sin(x)\, e^{-\cos(x)}$

$= e^{-\cos(x)}(\cos(x) + \sin^2(x))$

Recall the Pythagorean Identity $\cos^2(x) + \sin^2(x) = 1$.

$f''(x) = e^{-\cos(x)}(\cos(x) + 1 - \cos^2(x))$

1 mark

c. Hence, determine the values of x such that $f''(x) = f(x)$.

$e^{-\cos(x)}(\cos(x) + 1 - \cos^2(x)) = e^{-\cos(x)}$

$\cos(x) + 1 - \cos^2(x) = 1$

$0 = \cos^2(x) - \cos(x)$ [1 mark – working]

Let $u = \cos(x)$

$0 = u^2 - u$

$0 = u(u - 1)$

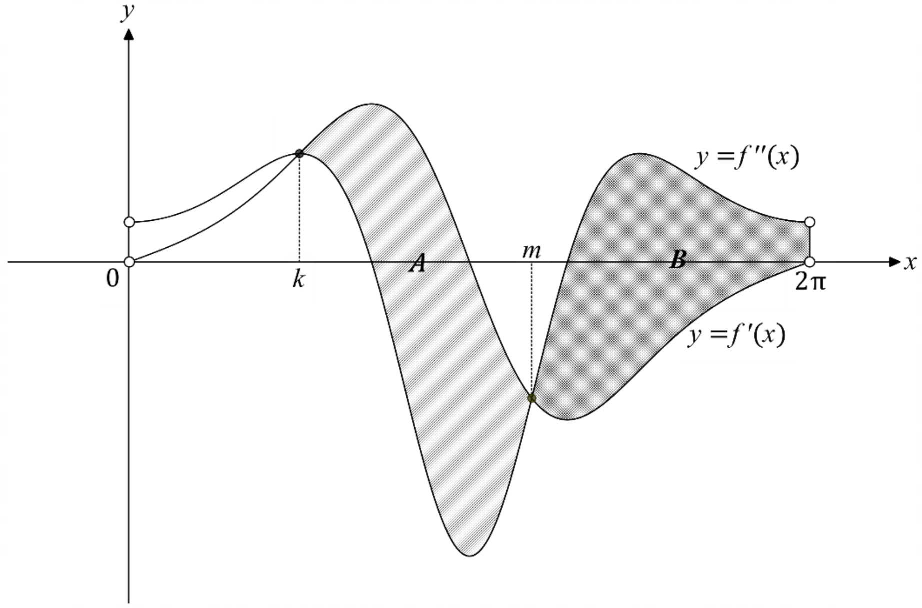
$u = 0$ or $u = 1$

$\cos(x) = 0$ or $\cos(x) = 1$ [1 mark – working]

$x = \frac{\pi}{2}, \frac{3\pi}{2}$ [1 mark – answer]

3 marks

The diagram below shows the graphs of $y = f'(x)$ and $y = f''(x)$, and their two intersections at $x = k$ and $x = m$. The region bounded by the graphs of $y = f'(x)$ and $y = f''(x)$ between $x = k$ and $x = m$ is denoted A. The region bounded by the graphs of $y = f'(x)$, $y = f''(x)$, and $x = 2\pi$ is denoted B. There is no overlapping area between region A and region B.



d. Use part **a** and part **c** to explain why $k = \frac{\pi}{2}$.

From part a, $f\left(\frac{\pi}{2}\right) = f'\left(\frac{\pi}{2}\right)$.

From part c, $f\left(\frac{\pi}{2}\right) = f''\left(\frac{\pi}{2}\right)$.

Hence, $f'\left(\frac{\pi}{2}\right) = f''\left(\frac{\pi}{2}\right)$

Hence the graph of $y = f'(x)$ and the graph of $y = f''(x)$ intersect at $x = \frac{\pi}{2}$.

Hence $k = \frac{\pi}{2}$.

1 mark

e. Given that region A has a bigger area than that of region B, determine the difference in area between region A and region B.

$\int_{\frac{\pi}{2}}^{m} f'(x) - f''(x)\, dx - \int_{m}^{2\pi} f''(x) - f'(x)\, dx$ [1 mark]

$= \int_{\frac{\pi}{2}}^{m} f'(x)\, dx - \int_{\frac{\pi}{2}}^{m} f''(x)\, dx - \int_{m}^{2\pi} f''(x)\, dx + \int_{m}^{2\pi} f'(x)\, dx$

$= \int_{\frac{\pi}{2}}^{2\pi} f'(x)\, dx - \int_{\frac{\pi}{2}}^{2\pi} f''(x)\, dx$ [1 mark]

$= [f(x)]_{\frac{\pi}{2}}^{2\pi} - [f'(x)]_{\frac{\pi}{2}}^{2\pi}$

$= \left[e^{-\cos(x)}\right]_{\frac{\pi}{2}}^{2\pi} - \left[e^{-\cos(x)} \sin(x)\right]_{\frac{\pi}{2}}^{2\pi}$

$= e^{-1} - e^{0} - (0 - e^{0} \cdot 1)$

$= e^{-1} - 1 + 1$

$= e^{-1}$ [1 mark – answer]

3 marks

END OF QUESTION AND ANSWER BOOK