Mathematical Methods Units 3 & 4

Written Examination 1 (Technology-free) 2025

Question and Answer Book

Student Name ________________

Teacher Name ________________

Reading Time: 15 minutes
Writing Time: 60 minutes

The total number of marks available is 40

Materials Supplied
- Question and Answer Book
- Formula Sheet

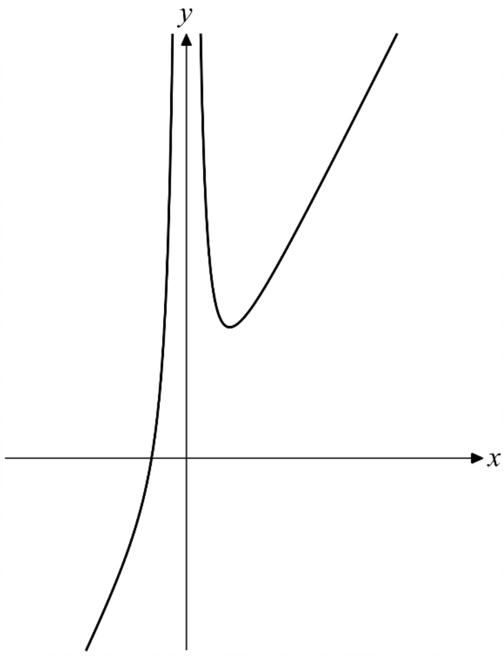
Instructions
- Students are **not** permitted to bring any technology (calculators or software), or notes of any kind, into the examination room.
- Students are **not** permitted to bring mobile phones and/or any unauthorised electronic devices into the examination room.

Instructions

- Answer all questions in the spaces provided.
- Write your responses in English.
- In questions where a numerical answer is required, an exact value must be given unless otherwise specified.
- In questions where more than one mark is available, appropriate working must be shown.
- Unless otherwise indicated, the diagrams in this book are not drawn to scale.

Question 1 (9 marks)

Consider function $f(x) = \frac{1}{x^2} + 2x$ defined over its maximal domain. The graph of $y = f(x)$ is shown in the diagram below.



a. Find the coordinates of the stationary point on the graph of $y = f(x)$.

2 marks

b. Newton's method is used to approximate the solution to the equation $f(x) = 0$.

i. If $x_0 = -2$, find the value of x_1.

1 mark

ii. State the values of x that are not allowed to be x_0.

1 mark

c. The bisection method is used to approximate the solution to the equation $f(x) = 0$. An algorithm written in pseudocode for the bisection method is shown below.

```
define f(x):
    return 1/x^2 + 2x

a ← ■
b ← ■
m ← (a+b)/2
count ← 0
while b − a > 2 × 0.0001
    if f(a) × f(m) < 0 then
        b ← m
    else
        a ← m
    end if
    [redacted]
    count ← count + 1
end while
print m, count
```

The initial values of a and b are not yet chosen, but it is assumed that $b > a$.
A line of code between "`end if`" and "$count \leftarrow count + 1$" is hidden.

i. Could the initial value of b be 0? If no, why? If yes, why and how should the initial value of a be chosen? Justify your answer with reasoning.

2 marks

ii. What is the hidden line of code between "`end if`" and "$count \leftarrow count + 1$"?

1 mark

d. Consider function $g(x) = -\frac{1}{x^2} + 2x$ defined over its maximal domain. Find the sequence of transformations that transform the graph of $y = f(x)$ to the graph of $y = g(x)$. Justify your answer with reasoning.

2 marks

Question 2 (10 marks)
A medical laboratory is testing a new rapid diagnostic test for a viral infection. The test has a 90% accuracy rate (probability of correctly identifying whether someone has the virus or not). Assume test results are independent of each other.

a. In a clinical trial, five patients are each tested once.

i. Find the probability that exactly two tests have correct results.

2 marks

ii. Find the most likely number of tests which have a correct result, correct to the nearest whole number.

1 mark

b. The laboratory decides to test patients one at a time until they get their first incorrect result, at which point they stop testing.

i. Find the probability that the first incorrect result occurs on the 3^{rd} test.

1 mark

ii. Given that the first two tests were all correct, find the probability that the first incorrect result occurs within the next two tests (i.e. on the 3^{rd} or 4^{th} test).

2 marks

c. A 'positive' test result is defined to indicate the patient is infected. A 'negative' test result is defined to indicate the patient is healthy.

To improve reliability, the laboratory implements a 'double-testing' protocol: each patient is tested twice, and the patient is diagnosed as infected only if both tests are positive. Otherwise, the patient is diagnosed as healthy.

Suppose a group contains six patients who are indeed infected and four patients who are indeed healthy. One patient is randomly selected from the group and double-tested.

i. Find the probability that an indeed infected patient is diagnosed as infected.

1 mark

ii. Find the probability that the randomly selected patient is diagnosed as healthy.

1 mark

iii. Given that the randomly selected patient is diagnosed as infected, find the probability that they are indeed infected.

2 marks

Question 3 (11 marks)

a. Find $\frac{d}{dx}(\cos(x)\log_e(\cos(x)))$, where e is Euler's number.

2 marks

b. Hence, show that $\int \sin(x)\log_e(\cos(x))\ dx = -\cos(x)\log_e(\cos(x)) + \cos(x) + c$, where c is a constant.

1 mark

c. Hence, show that $\int_0^{\frac{\pi}{3}} \sin(x)\log_e(\cos(x))\ dx = \frac{1}{2}\log_e(2) - \frac{1}{2}$.

2 marks

Let $f(x) = \sin(x)\log_e(\cos(x))$.

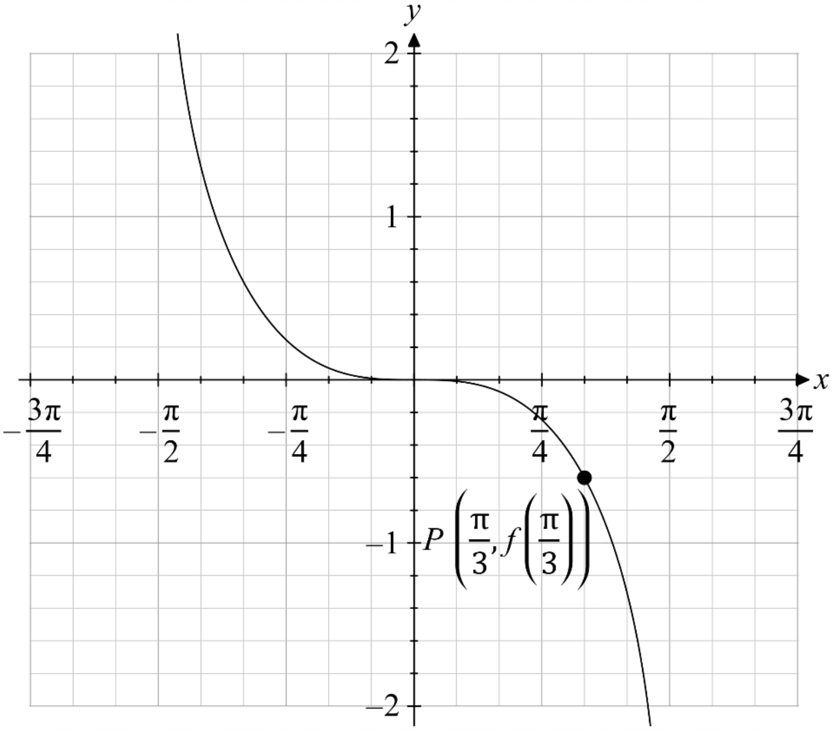
d. On the graph of $y = f(x)$, there is an asymptote at $x = \frac{\pi}{2}$. Explain why there is an asymptote at $x = \frac{\pi}{2}$.

1 mark

e. For $x \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$, by considering individual components of $f(x)$, prove that $f(-x) = -f(x)$.

1 mark

The graph of $y = f(x)$ for $x \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$ is shown in the diagram below. The point P has coordinates $\left(\frac{\pi}{3}, f\left(\frac{\pi}{3}\right)\right)$.



f. On the diagram above, clearly draw the graph of $y = -f(x)$ for $x \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$. Label the image of point P as P^*.

2 marks

g. Hence, find the area bounded by the curves of $y = f(x)$ and $y = -f(x)$, and the lines $x = -\frac{\pi}{3}$ and $x = \frac{\pi}{3}$.

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2 marks

Question 4 (10 marks)

Consider function

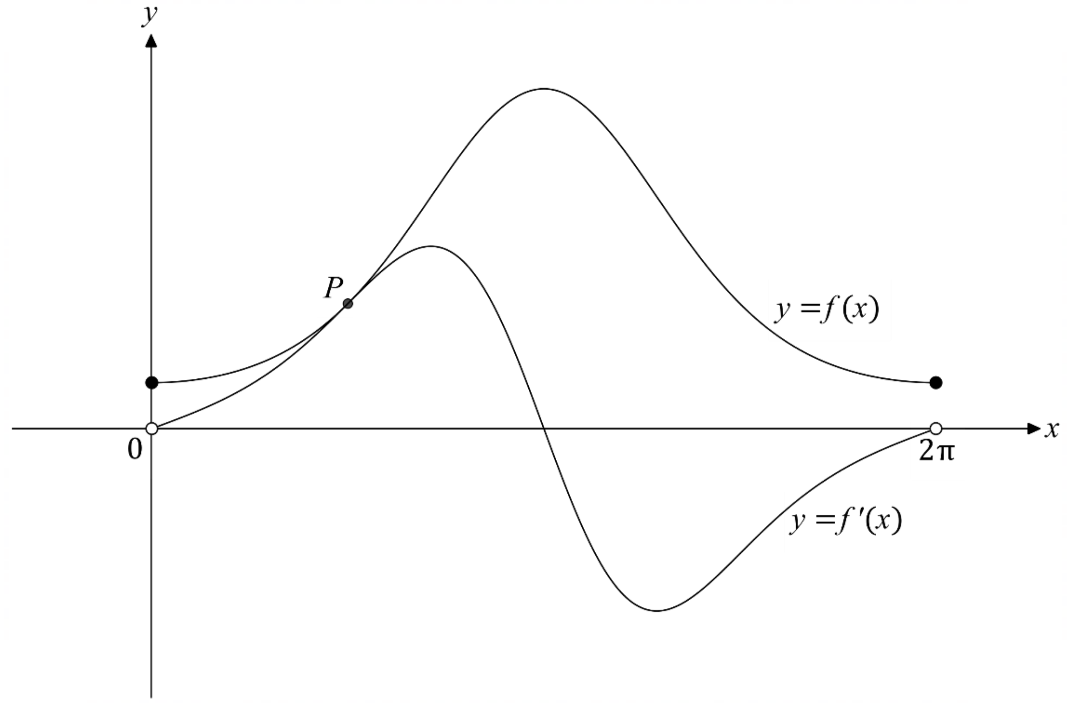
$$f: [0, 2\pi] \to \mathbb{R}, f(x) = e^{-\cos(x)}$$

and its derivative function

$$f': (0, 2\pi) \to \mathbb{R}, f'(x) = e^{-\cos(x)}\sin(x)$$

where e is Euler's number.

The diagram below shows the graphs of $y = f(x)$ and $y = f'(x)$. The two graphs touch at point P.



a. Determine the x-coordinate of point P.

2 marks

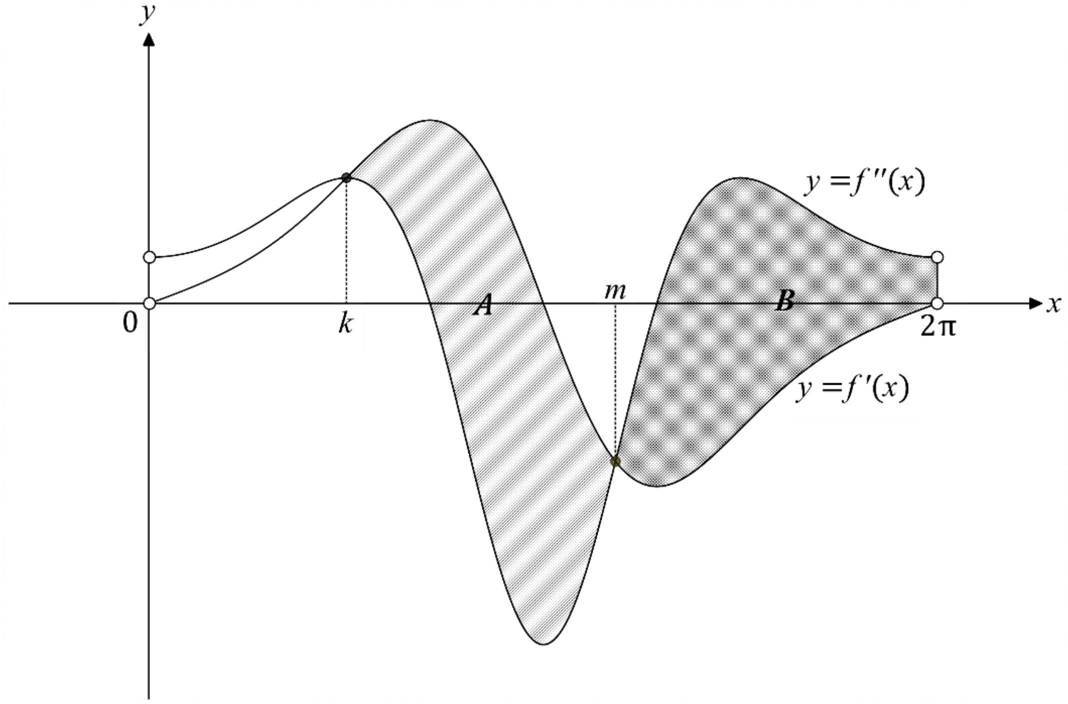
b. Show that $f''(x) = e^{-\cos(x)}(\cos(x) + 1 - \cos^2(x))$. Clearly state any trigonometric identity you use.

1 mark

c. Hence, determine the values of x such that $f''(x) = f(x)$.

3 marks

The diagram below shows the graphs of $y = f'(x)$ and $y = f''(x)$, and their two intersections at $x = k$ and $x = m$. The region bounded by the graphs of $y = f'(x)$ and $y = f''(x)$ between $x = k$ and $x = m$ is denoted A. The region bounded by the graphs of $y = f'(x)$, $y = f''(x)$, and $x = 2\pi$ is denoted B. There is no overlapping area between region A and region B.



d. Use part **a** and part **c** to explain why $k = \frac{\pi}{2}$.

1 mark

e. Given that region A has a bigger area than that of region B, determine the difference in area between region A and region B.

3 marks

END OF QUESTION AND ANSWER BOOK