bigWU

Mathematical Methods Units 3 & 4

Written Examination 2 (Technology-active) 2025

Question and Answer Book

Student Name ________________

Teacher Name ________________

Reading Time: 15 minutes
Writing Time: 120 minutes

The total number of marks available is 80

Approved Materials
- Protractors, set squares, and aids for curve sketching
- One bound reference
- One approved CAS calculator or CAS software, and one scientific calculator

Materials Supplied
- Question and Answer Book
- Formula Sheet
- Multiple-choice Answer Sheet

Instructions
- Follow the instructions on your Multiple-choice Answer Sheet.
- At the end of the examination, place your Multiple-choice Answer Sheet inside the front cover of this book.

Students are **not** permitted to bring mobile phones and/or any unauthorised electronic devices into the examination room.

Section A – Multiple-choice Questions

Instructions

- Answer all questions in pencil on your Multiple-choice Answer Sheet.
- Choose the response that is correct for the question.
- A correct answer scores 1; an incorrect answer scores 0.
- Marks will not be deducted for incorrect answers.
- No marks will be given if more than one answer is completed for any question.
- Unless otherwise indicated, the diagrams in this book are not drawn to scale.

Question 1

Consider function $f(x) = \frac{1}{x^2}$. What is the central difference approximation of $f'(2)$ where h represents the horizontal distance from $x = 2$ to the left and to the right, and $h = 0.2$, correct to three decimal places?

A. -0.230
B. -0.250
C. -0.284
D. -0.255

Question 2

The graph of a cubic polynomial goes through the point $(-2, 0)$ and it has a local maximum at $(1, 3)$. What is a possible rule of the cubic polynomial?

A. $y = x^3 + 3x^2 - 5x$
B. $y = -\frac{2}{3}x^3 - \frac{4}{5}x^2 + 6$
C. $y = \frac{1}{2}x^3 - \frac{1}{3}x^2 + \frac{1}{6}x + 2$
D. $y = -\frac{5}{6}x^3 - \frac{1}{3}x^2 + \frac{19}{6}x + 1$

Question 3

Consider function $f: [-\pi, \pi] \to \mathbb{R}, f(x) = \tan(nx)$, where $n \in \mathbb{Z}^+$. How many asymptotes are there in the graph of $y = f(x)$?

A. n
B. $n + 1$
C. $\frac{n}{2}$
D. $2n$

Question 4

Consider function $f: [-5, a] \to \mathbb{R}, f(x) = x^2 - 4x + 5$, where a is a constant. What is the largest set of values of a for which the inverse function f^{-1} exists?

A. $(-5, 2]$
B. $\{2\}$
C. $[-5, 2]$
D. $(-5, 2)$

Question 5

Consider the system of equations

$$\begin{cases} mx + (m-3)y = 2m \\ (m+1)x + 2y = m + 4 \end{cases}$$

where m is a constant. What are the values of m for which there is no solution for (x, y)?

A. $\left(-\sqrt{7} + 2, \sqrt{7} + 2\right)$
B. $\left[-\sqrt{7} + 2, \sqrt{7} + 2\right]$
C. $\left\{-\sqrt{7} + 2, \sqrt{7} + 2\right\}$
D. $\mathbb{R} \setminus \left\{-\sqrt{7} + 2, \sqrt{7} + 2\right\}$

Question 6

In a group of 60 students, 37 play basketball, 36 play soccer, and 8 do not play either basketball or soccer. How many students play both basketball and soccer?

A. 21
B. 20
C. 22
D. 19

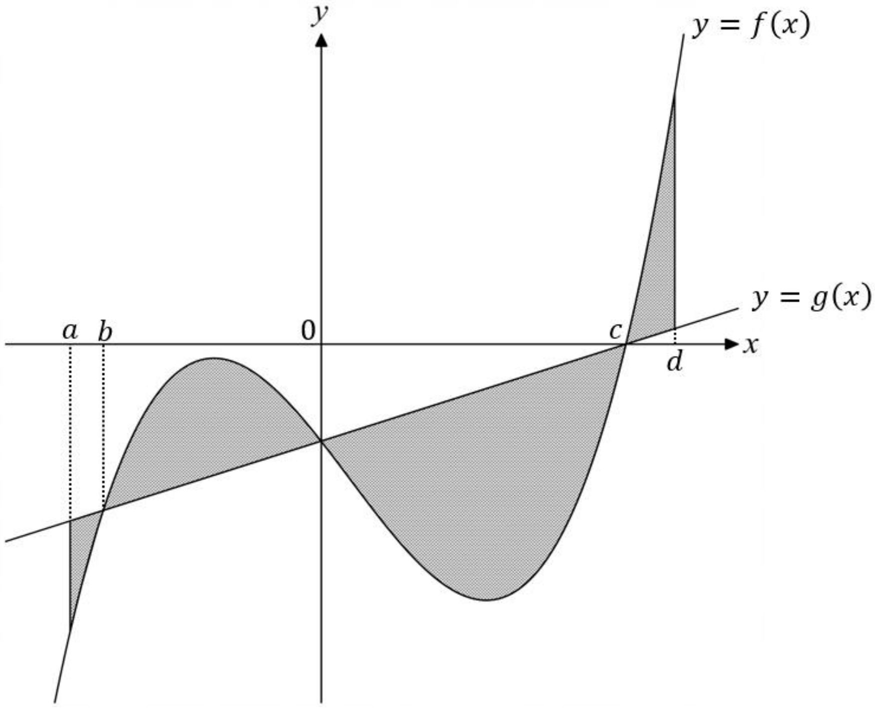
Question 7

Consider functions $f: \mathbb{R} \setminus \{0\} \to \mathbb{R}, f(x) = \frac{1}{x}$ and $g: D \to \mathbb{R}, g(x) = \sin(x)$, where D is the domain of g. What is a possible set for D for which the composite function $(f \circ g)(x)$ exists?

A. $[0, \pi]$
B. $\left(\frac{\pi}{2}, \frac{3\pi}{2}\right)$
C. $(0, \pi)$
D. $\left[\frac{\pi}{2}, \frac{3\pi}{2}\right]$

Question 8

Consider the graphs of $y = f(x)$ and $y = g(x)$ and the regions shaded in the diagram below.



Which of the following gives the total area of the shaded regions?

A. $\int_a^b f(x) - g(x)\,dx + \int_0^b f(x) - g(x)\,dx + \int_c^0 f(x) - g(x)\,dx - \int_c^d g(x) - f(x)\,dx$

B. $\int_b^a g(x) - f(x)\,dx - \int_b^0 f(x) - g(x)\,dx + \int_0^c f(x) - g(x)\,dx + \int_c^d f(x) - g(x)\,dx$

C. $\int_a^b g(x) - f(x)\,dx + \int_b^0 f(x) - g(x)\,dx + \int_0^c g(x) - f(x)\,dx - \int_c^d f(x) - g(x)\,dx$

D. $-\int_a^b g(x) - f(x)\,dx - \int_b^0 f(x) - g(x)\,dx + \int_0^c g(x) - f(x)\,dx + \int_c^d g(x) - f(x)\,dx$

Question 9

Mia's marks in Year 10 assessments are shown. The scores for each subject were normally distributed.

	Mia's mark	*Year 10 mean*	*Year 10 standard deviation*
English	70	65	7
Mathematics	85	71	10
Science	80	75	8
Humanities	65	72	9

In which subject did Mia perform the best in comparison with the rest of Year 10?

A. English
B. Mathematics
C. Science
D. Humanities

Question 10

A quality control process tests components, where each component has a probability p of passing the test, independently of other components. A batch of n components is tested. It is known that the probability that all components pass is 0.0625 and the probability that exactly $(n-1)$ components pass is 0.25. What are the values of n and p?

A. $n = 6, p = 0.32$
B. $n = 3, p = 0.1$
C. $n = 5, p = 0.4$
D. $n = 4, p = 0.5$

Question 11

A manufacturing company produces computer chips. From historical data, it is known that 15% of computer chips are defective. The quality control manager takes a sample of n chips to inspect. Let $\hat{p}$ be the random variable representing the proportion of chips in the sample that are defective. If the quality control manager wants the standard deviation of $\hat{p}$ to be less than 0.02, what is the minimum value of n?

A. 318
B. 319
C. 320
D. 300

Question 12

The parabola $y = (x-2)^2 - 3$ is translated two units in the positive direction of the vertical axis, then reflected in the horizontal axis, then dilated by a factor of two from the vertical axis, then translated 1 unit in the negative direction of the horizontal axis. What is the equation of the resulting parabola?

A. $y = -\frac{1}{4}(x-6)^2 + 5$
B. $y = -\frac{1}{4}(x-3)^2 + 1$
C. $y = -4\left(x-\frac{3}{2}\right)^2 + 1$
D. $y = -4\left(x-\frac{1}{2}\right)^2 + 5$

Question 13

Given that $\frac{d}{dx}(xe^{-x}) = e^{-x} - xe^{-x}$, where e is Euler's number, what is a possible antiderivative of xe^{-x}?

A. $\int e^{-x}\,dx - xe^{-x}$
B. $e^{-x} - xe^{-x}$
C. $e^{-x} + xe^{-x}$
D. $-\int e^{-x}\,dx - xe^{-x}$

Question 14

What is the maximal domain of the function $f(x) = \log_e\left(\sqrt{x^2 - 1}\right)$?

A. $[-1, 1]$
B. $\mathbb{R} \setminus (-1, 1)$
C. $\mathbb{R} \setminus [-1, 1]$
D. $(-1, 1)$

Question 15

An airline estimates that the probability of any flight being delayed is 0.32. The airline wishes to take a sample of n flights and find the proportion of delayed flights. If the airline wishes to achieve a margin of error less than 0.03 for a 95% confidence interval, what is the minimum sample size required?

A. 927
B. 928
C. 929
D. 930

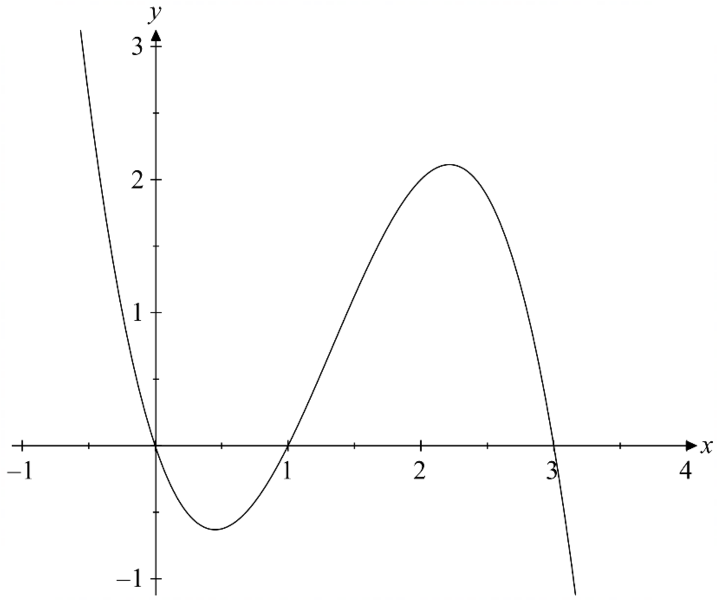
Question 16

The area bounded by the graph of $y = -x(x - 5)$ and the x-axis is estimated using the trapezium rule. If the difference between the actual area and the estimated area needs to be less than 1, what is the minimum number of trapeziums required?

A. 5
B. 6
C. 4
D. 7

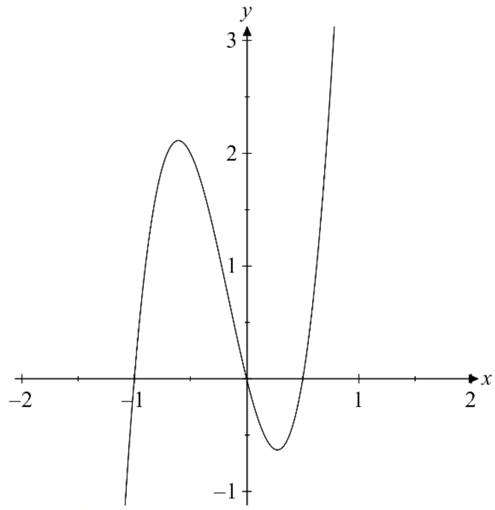
Question 17

The diagram below shows the graph of $y = f(x)$.

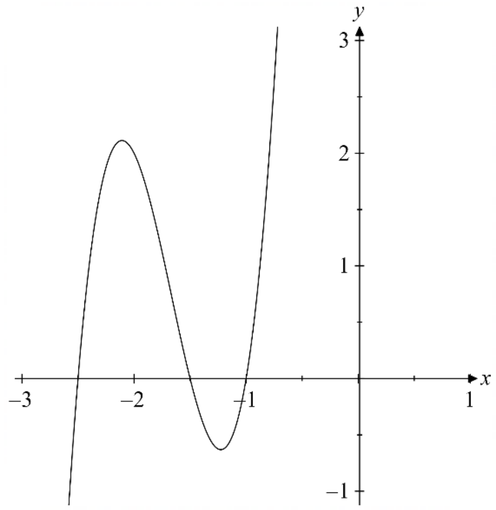


Which of the following best represents the graph of $y = f(-2x - 1)$?

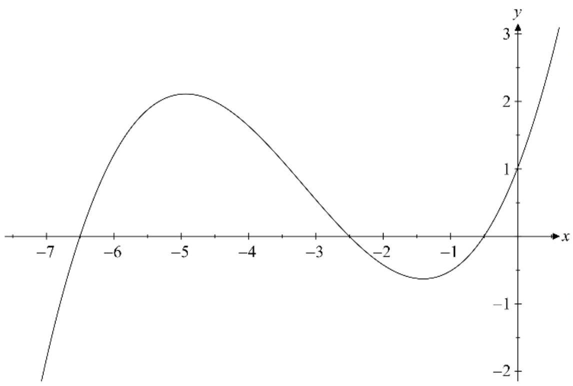
A.



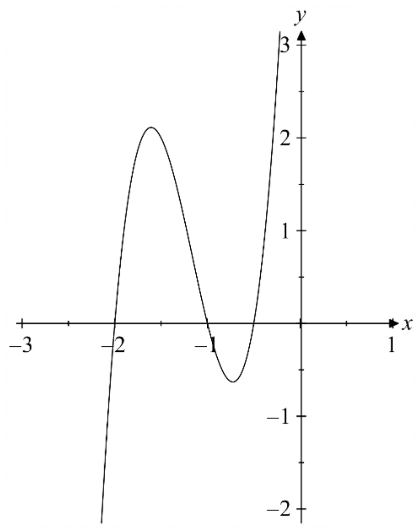
B.



C.

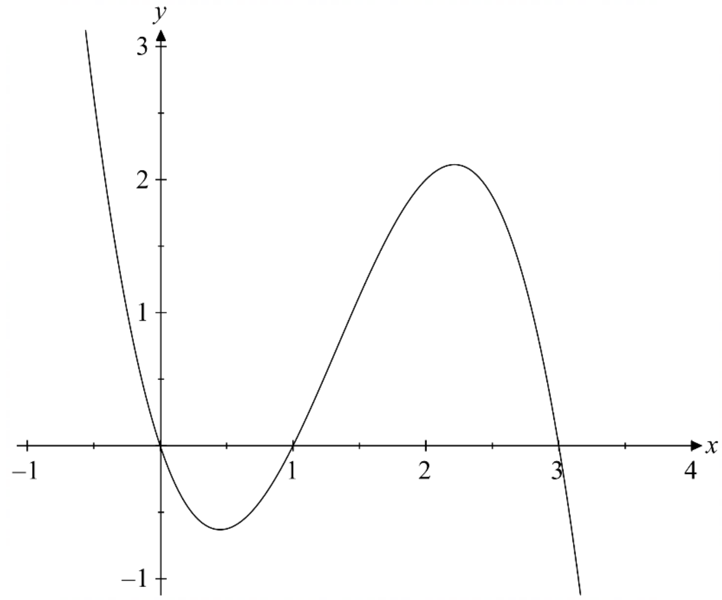


D.



Question 18

The diagram below shows the graph of $y = f(x)$, where the domain of f is $\mathbb{R}$.



Which one of the following functions is defined over the domain $\mathbb{R}$?

A. $\sqrt{f(x)}$

B. $\log_e(f(x))$

C. $e^{f(x)}$

D. $\frac{1}{f(x)}$

Question 19

A bag contains 3 red marbles and 5 white marbles. Jo randomly selects two marbles at the same time from the bag. Given that one of the marbles that Jo has selected is red, what is the probability that the other marble selected is white?

A. $\frac{5}{6}$

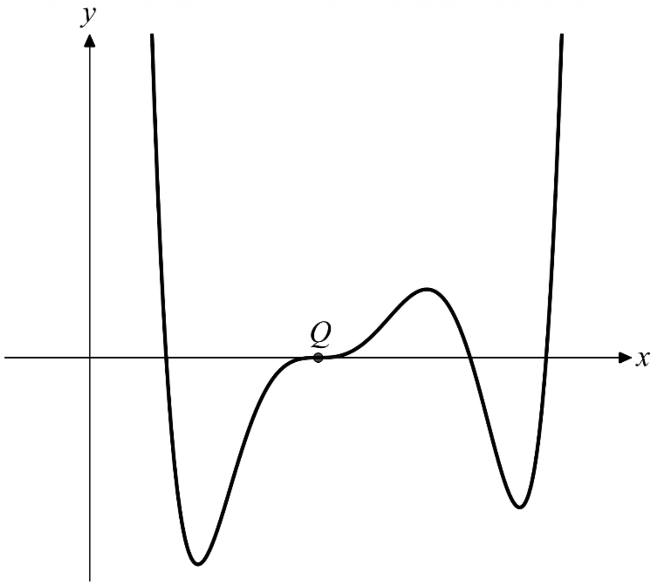
B. $\frac{7}{10}$

C. $\frac{1}{5}$

D. $\frac{1}{10}$

Question 20

The diagram below shows the graph of $y = f(x)$.



The point Q is a stationary point of inflection.

Let $A(x) = \int_0^x f(t)\, dt$.

How many points of inflection does the graph of $y = A(x)$ have?

A. 2

B. 3

C. 4

D. 5

END OF SECTION A

Section B

Instructions

- Answer all questions in the spaces provided.
- Write your responses in English.
- In questions where a numerical answer is required, an exact value must be given unless otherwise specified.
- In questions where more than one mark is available, appropriate working must be shown.
- Unless otherwise indicated, the diagrams in this book are not drawn to scale.

Question 1 (17 marks)

Consider functions $f(x) = \log_e(x+2)$ and $g(x) = e^{-x} + 1$, where e is Euler's number.

a. State the domain and range of $f(x)$ and $g(x)$.

2 marks

b. With justification and reasoning, show that the function $f\big(g(x)\big)$ exists.

1 mark

Let $h(x) = f\big(g(x)\big)$.

c. State the rule of $h(x)$.

1 mark

d. Find the exact value of x for which $h(x) = \log_e(5)$.

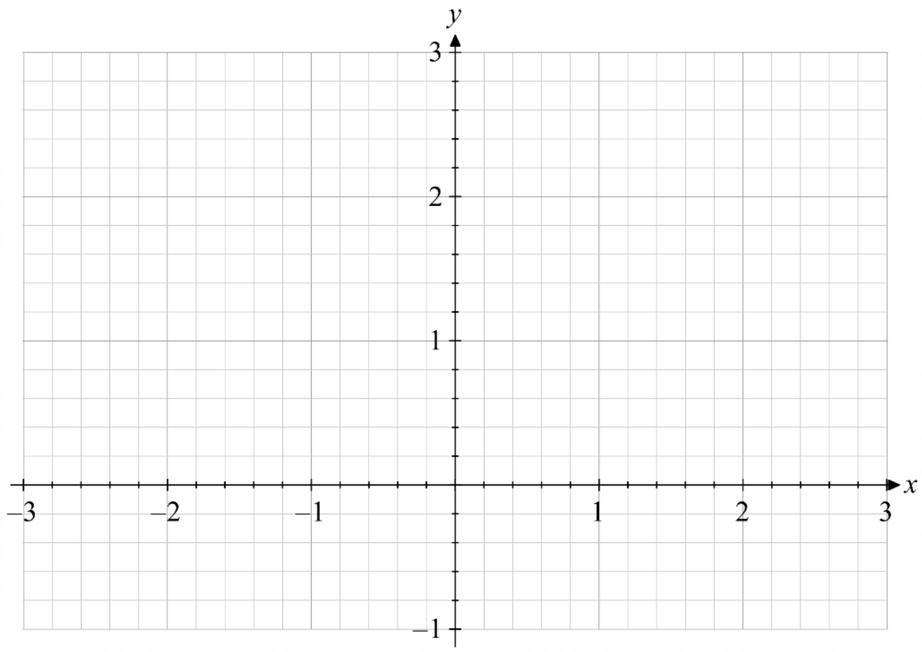
2 marks

e. By first evaluating $\lim\limits_{x \to \infty} g(x)$, evaluate $\lim\limits_{x \to \infty} h(x)$.

2 marks

f. On the axes provided below, sketch the graph of $y = h(x)$, labelling coordinates of y-axis intercept and the rule of asymptote.

4 marks



A marine biologist models the oxygen concentration (in mg/L) at a fixed point in a water tank after some oxygen is pumped into the water once, using the function

$$C(t) = h(t-4) + 8, \qquad t \geq 0$$

where t represents time in hours and $t = 0$ corresponds to the instant when oxygen is pumped into the water.

g. Find the oxygen concentration in mg/L at the instant when oxygen is pumped into the water, correct to two decimal places.

1 mark

h. The oxygen concentration stabilises after a long time, meaning it stays at a constant level. Find the eventual oxygen concentration after a long time in mg/L, correct to two decimal places.

1 mark

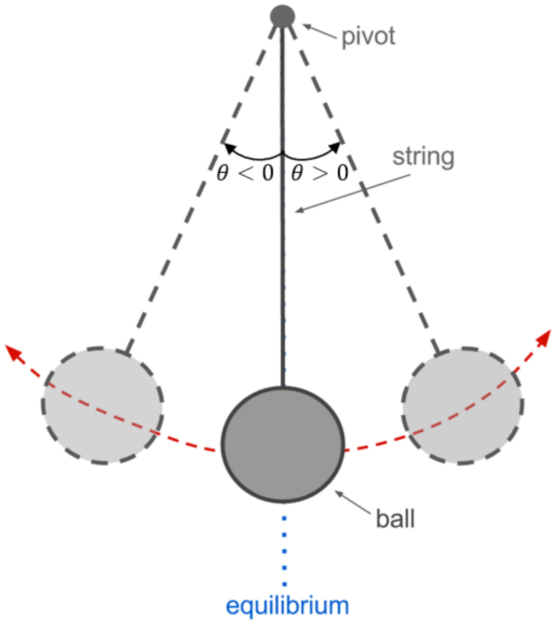
i. There is a type of fish in the water tank which will struggle to survive at an oxygen concentration level below 9.50 mg/L. Find the value of t beyond which the fish will struggle to survive, correct to two decimal places.

1 mark

j. There is a type of fish in the water tank for which the optimal oxygen concentration level is between 10 and 11 mg/L inclusive. Find the time duration in which the oxygen concentration level is optimal for the fish, correct to the nearest minute.

2 marks

Question 2 (16 marks)
A pendulum is depicted in the diagram below. It consists of a rigid string, a pivot (fixed point) connected at the top of the string, and a ball connected at the bottom of the string. The equilibrium refers to the situation where the ball is not moving, and the ball is stationary directly (vertically) below the pivot. When the ball is displaced from the equilibrium position then allowed to travel freely, it travels in a circular path around the pivot point, and because of gravity, it swings left and right, going past the equilibrium position back and forth. Due to air resistance, the extent of the ball's swing will decrease over time, until it comes to a complete stop.



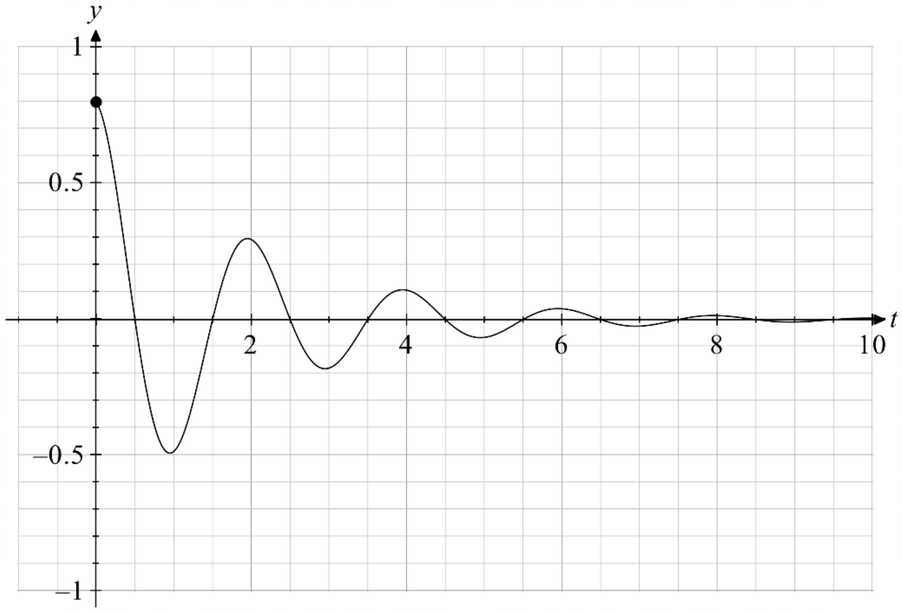
The angle θ (in radians) between the string and the equilibrium position is measured. When the ball is to the right of the equilibrium, θ is defined to be positive. When the ball is to the left of the equilibrium, θ is defined to be negative.

Throughout this question, e is Euler's number, t represents time in seconds, and $t = 0$ corresponds to the instant right after the ball is displaced from the equilibrium position.

a. The angle θ is modelled by the function

$$\theta(t) = 0.8e^{-0.5t}\cos(\pi t), \qquad t \geq 0$$

The graph of $y = \theta(t)$ is shown in the diagram below.



i. State the initial angle between the string and the equilibrium position.

1 mark

ii. Find the time in seconds when the ball <u>first</u> returns to the equilibrium position.

1 mark

iii. State $\theta'(2)$.

1 mark

iv. Find the value of θ when the ball is furthest away from the equilibrium and the ball is to the left of the equilibrium position, correct to four decimal places.

2 marks

v. State the value of t at which θ is changing at the highest rate, correct to two decimal places.

1 mark

vi. Find the average rate of change between the first local minimum and the first local maximum in the graph of $y = \theta(t)$, correct to two decimal places.

2 marks

vii. Find the average value of θ between $t = 0$ and $t = 0.5$, correct to four decimal places.

2 marks

viii. The pendulum is considered to have "effectively stopped" when θ is permanently between -0.1 and 0.1 exclusive. Find the earliest time after which the pendulum is considered to have "effectively stopped", correct to two decimal places.

3 marks

b. A different pendulum is set up and its behaviour is modelled by the function

$$\theta(t) = ae^{-bt}\cos(ct), \qquad t \geq 0$$

where a, b, and c are constants.

The following are observed:

- The initial angle between the string and the equilibrium position is 0.6 radians.
- The first maximum angle to the left of equilibrium occurs at $t = 1.8$.
- The first maximum angle to the left of equilibrium is -0.25 radians.

i. Show that $a = 0.6$.

1 mark

ii. If $0 < b < 1$ and $0 < c < 2$, find the values of b and c, correct to two decimal places.

2 marks

Question 3 (10 marks)
The performance of kicking set shots of Australian Rules Football (AFL) players is analysed. Historical data shows that the distance from goal for set shots, D metres can be modelled by a normal distribution with mean 35 metres and standard deviation 12 metres.

a. State the probability that a randomly selected set shot is taken from a distance between 25 and 40 metres, correct to four decimal places.

1 mark

b. State the distance above which 80% of set shots are taken, correct to one decimal place.

1 mark

Given that a set shot is taken from a distance lower than 30 metres, the probability of making the shot is 0.65, independent of other set shot attempts.

c. A sample of 12 set shots taken from a distance lower than 30 metres are randomly selected. Let X be the number of made shots in this sample.

i. Find the probability that the number of made shots in the sample is between 6 and 9 inclusive, correct to four decimal places.

2 marks

ii. Find $E(X)$ and $sd(X)$, correct to two decimal places.

2 marks

d. The probability of any player making any set shot is 0.53, regardless of distance from goal. Given that a set shot is missed, find the probability that the set shot is taken from a distance greater than 30 metres.

4 marks

Question 4 (17 marks)

Consider function $f: \mathbb{R} \to \mathbb{R}, f(x) = x^3 - 6x^2 + 12x - 5$.

a. State coordinates of all stationary point(s) on the graph of $y = f(x)$, and state their nature.

2 marks

b. Hence or otherwise, justify the existence of the inverse function $f^{-1}(x)$.

1 mark

c. Express the rule of function f in the form $f(x) = (x + a)^3 + b$, where a and b are constants.

1 mark

Consider function $g: (-\infty, 2] \rightarrow \mathbb{R}, g(x) = x^3 - 6x^2 + 12x - 5$.

d. State the inverse function $g^{-1}(x)$. Express your answer in full function notation.

3 marks

e. The graph of $y = g(x)$ is reflected in the line $y = x$, followed by a translation of 2 units in the positive direction of the x-axis and a translation of k units in the positive direction of the y-axis. The resulting graph has an end point at $(5, 6)$. State the value of k.

1 mark

Consider function $h: D \rightarrow \mathbb{R}, h(x) = x^2 + 1$, where set D is the domain of h.

f. Find the maximum domain of h for which the function $g^{-1}\left(h(x)\right)$ is defined.

2 marks

g. Consider equation $f^{-1}(x) + f^{-1}(24 - x) = 6$. This will be referred to as '*the equation*' in question **g**. We will attempt to solve *the equation* $f^{-1}(x) + f^{-1}(24 - x) = 6$.

i. Show that if $x = c$ is a solution to *the equation*, then $x = 24 - c$ is also a solution to *the equation*.

1 mark

Let $u = f^{-1}(x)$ and let $v = f^{-1}(24 - x)$.

ii. Re-write *the equation* using u and v.

1 mark

iii. Express $f(u)$ and $f(v)$ in terms of x.

1 mark

iv. Hence or otherwise, show that $f(u) + f(6 - u) = 24$.

1 mark

v. State the solutions for u in $f(u) + f(6 - u) = 24$.

1 mark

vi. Hence or otherwise, find the solutions to *the equation* $f^{-1}(x) + f^{-1}(24 - x) = 6$.

2 marks

END OF QUESTION AND ANSWER BOOK